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Strongly Unique Best Approximation in Banach Spaces

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1. INTRODUCTION AND PRELIMNARIES

Let X be a linear closed subspace of the real Banach space Y. An element z in X is called a best approximation to an element y in Y if

$$\|y-z\| \leqslant \|y-x\|$$

for all x in X. Following Papini and Singer [14], we call an element z in X a best coapproximation to an element y in Y if

$$\|z-x\| \leqslant \|y-x\|$$

for all x in X. This kind of "approximation" has been introduced by Franchetti and Furi [8].

DEFINITION 1.1. An element z in X is called a strongly unique best approximation to an element y in Y if there exist a positive number K and a strictly increasing continuous function $\varphi\colon [0, +\infty) = \mathbb{R}_+ \to \mathbb{R}_+$; $\varphi(0) = 0$, such that

$$\varphi(\|y-z\|) \leqslant \varphi(\|y-x\|) - K\varphi(\|z-x\|) \tag{1.1}$$

for all x in X.

From the definition it immediately follows that a strongly unique best approximation z in X to an element y in Y is a unique best approximation in X to y. Moreover, if $K = K(y) \geqslant 1$ then z is also a unique best coapproximation in X to y. When X is a Haar subspace of $C(B)$, the space of continuous real valued functions on a compact Hausdorff space B with the supremum norm, Newman and Shapiro [11] have shown that to every y in $C(B)$ there exists a strongly unique best approximation in X with

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$\varphi(s)=s$ and $0<K\leqslant 1$. More recently, a number of papers [3–5, 12, 13, 15] have examined the largest constant K in (1.1) as a functional of y and characterized a strongly unique best approximation to y in $C(B)$. On the other hand, it is well known [18] that a strongly unique best approximation with $\varphi(s)=s$ need not exist for every y in Y when Y is a smooth space (in particular an L_p space). Therefore, it would be important to know in this case whether there exists a strongly unique best approximation to every y in Y in the sense of Definition 1.1. If so, it would be desirable to give formulae for the constant K and the function φ in (1.1). In this paper we propose a unified approach to deal with these problems for a linear closed subspace X of a real Banach space Y. The approach consists of using the following theorem due to Leżański [9].

THEOREM 1.1. *Let $f\colon X\to\mathbb{R}$ be a functional satisfying the following two conditions*:

(i) *There exists a nondecreasing continuous function $d\colon\mathbb{R}_+\to\mathbb{R}_+$ such that $\|x_i\|\leqslant r$ ($x_i\in X$, $i=1,2$) implies that*

$$|f(x_1)-f(x_2)|\leqslant d(r)\,\|x_1-x_2\|;$$

(ii) *For any $t\in(0,1)$ and $x,h\in X$ we have*

$$\begin{aligned} g(t;x,h) &:= tf(x+h)+(1-t)f(x)-f(x+th)\\ &\geqslant c(t,\|h\|), \end{aligned}$$

where

$$c(t,s)=tb((1-t)\,s)+(1-t)\,b(ts),\qquad 0\leqslant t\leqslant 1\quad\text{and}\quad s\geqslant 0,$$

$$b(s)=\int_0^s a(t)\,dt,\qquad s\geqslant 0,$$

and $a\colon\mathbb{R}_+\to\mathbb{R}_+$ is a continuous strictly increasing function such that

$$a(0)=0\qquad\text{and}\qquad\lim_{s\to+\infty}a(s)=+\infty.$$

Then there exists a unique element $z\in X$ such that

$$f(z)\leqslant f(x)\qquad\text{and}\qquad\|x-z\|\leqslant b^{-1}(f(x)-f(z))$$

for every x in X.

The main results presented in this paper are strong unicity theorems for L_p ($p\geqslant 2$) spaces and for abstract spline approximation. In particular, this solves the following problem posed by Dunham [7, Problem 41]: What is the counterpart of strong uniqueness for L_p approximation?

It is interesting to note that Angelos and Egger [1] have introduced recently some other notion of strong uniqueness in L_p spaces. This strong uniqueness is strictly a local property, unlike the Newman and Shapiro concept of strong uniqueness in $C(B)$ and its generalization given in Definition 1.1.

2. Strong Unicity for Hilbert Spaces

In this section we assume that X is a linear closed subspace of the real Hilbert space Y. Then the following theorem holds.

THEOREM 2.1. *For an element y in Y there exists a unique element z in X such that*

$$\|y-z\|^2 \leqslant \|y-x\|^2 - \|z-x\|^2 \tag{2.1}$$

for all x in X.

Proof. Let us set

$$f(x) = \|y-x\|^2, \qquad a(s) = 2s,$$

and

$$d(r) = 2(r + \|y\|)$$

into Theorem 1.1. Then

$$b(s) = s^2 \qquad \text{and} \qquad c(t, s) = t(1-t)\, s^2.$$

Moreover, we have

$$|f(x_1) - f(x_2)| = |(2y - x_2 - x_1, x_2 - x_1)| \leqslant d(r)\, \|x_1 - x_2\|$$

for all $x_i \in X$ ($\|x_i\| \leqslant r$, $i = 1, 2$), and

$$g(t; x, h) = t(1-t)\, \|h\|^2 = c(t, \|h\|)$$

for any $t \in (0, 1)$ and $x, h \in X$. Therefore, the assumptions (i) and (ii) in Theorem 1.1 are satisfied in this case. Hence by using this theorem we conclude that there exists a unique element z in X such that

$$f(z) \leqslant f(x) \qquad \text{and} \qquad \|x - z\| \leqslant (f(x) - f(z))^{1/2}, \qquad x \in X.$$

This completes the proof. ▮

The theorem shows that there exists a strongly unique best approximation z in X to every $y \in Y$ in the sense of Definition 1.1, $\varphi(s)=s^2$ and $K=1$. Hence a strongly unique best approximation z in X to an element y in Y is both a unique best approximation and a coapproximation in X to y. Thus, we can define a linear projection P of Y onto X by setting $Py=z$. If $x=0$ is inserted into (2.1) then one can derive the corollary.

COROLLARY 2.1. *For every y in Y we have*

$$\|Py\|^2 \leqslant \|y\|^2 - \|y-Py\|^2. \tag{2.2}$$

As an immediate consequence of (2.2) we obtain the following well-known result.

COROLLARY 2.2. *The projection P is a linear norm 1 projection of Y onto X and $\|Py\| = \|y\|$ iff $y \in X$.*

3. STRONG UNICITY FOR SPLINES IN HILBERT SPACES

Throughout this section it is assumed that T is a bounded linear operator on a real Banach space Y to a real Hilbert space Y_1. Moreover, let X be a linear closed subspace of Y such that the linear subspace $X_1 = T(X)$ is closed in Y_1 and

$$X \cap \ker T = \{0\}.$$

Clearly, these assumptions ensure that the operator $T_0 = T|_X$ has a bounded linear inverse $T_0^{-1}: X_1 \to X$. An element $\sigma = y - z$ $(z \in X)$ is called a spline approximation to an element y in Y if

$$\|T\sigma\| \leqslant \|Ty - Tx\|$$

for all x in X.

Remark 3.1. If G is a subset of the conjugate space Y^* of Y and

$$X = \bigcap_{g \in G} \ker g$$

then the above definition of a spline approximation σ to an element y of Y reduces to the usual definition of a (T, G)-spline interpolant σ to y introduced by Atteia [2] (cf. also de Boor [6]).

THEOREM 3.1. *For an element y in Y there exists a unique element $\sigma = y - z$ $(z \in X)$ such that*

$$\|T\sigma\|^2 \leqslant \|Ty - Tx\|^2 - \|T_0^{-1}\|^{-2} \|z - x\|^2 \tag{3.1}$$

for all x in X.

Proof. If we insert

$$f(x) = \|Ty - Tx\|^2, \qquad a(s) = 2\,\|T_0^{-1}\|^{-2}\, s,$$

and

$$d(r) = 2\,\|T\|^2 (r + \|y\|)$$

into Theorem 1.1 then

$$b(s) = \|T_0^{-1}\|^{-2}\, s^2 \qquad \text{and} \qquad c(t, s) = t(1-t)\,\|T_0^{-1}\|^{-2}\, s^2.$$

Moreover, we have

$$\begin{aligned} |f(x_1) - f(x_2)| &= |(2Ty - Tx_2 - Tx_1, Tx_2 - Tx_1)| \\ &\leqslant d(r)\,\|x_1 - x_2\| \end{aligned}$$

for all $x_i \in X$ $(\|x_i\| \leqslant r,\ i = 1, 2)$ and

$$g(t; x, h) = t(1-t)\,\|Th\|^2 \geqslant c(t, \|h\|)$$

for any $t \in (0, 1)$ and $x, h \in X$. Hence we can apply Theorem 1.1 to complete the proof. ▌

The theorem shows that there exists a strongly unique best approximation Tz in X_1 to every Ty $(y \in Y)$ in the sense of Definition 1.1, $\varphi(s) = s^2$ and $K = \|T_0^{-1}\|^{-2}$. In other words, we can say that the element $\sigma = y - z$ is a strongly unique spline approximation in X to y. Clearly, it is a unique spline approximation in X to y. Now, let a linear spline projection P be defined by $Py = \sigma$, $y \in Y$. Then setting $x = 0$ into (3.1) we immediately obtain

COROLLARY 3.1. *For every y in Y we have*

$$\|y - Py\|^2 \leqslant \|T_0^{-1}\|^2 (\|Ty\|^2 - \|TPy\|^2). \tag{3.2}$$

Let us note that the inequality (3.2) yields the well-known [6] estimates

$$\|I - P\| \leqslant \|T_0^{-1}\|\, \|T\|$$

and

$$\|P\| \leqslant 1 + \|T_0^{-1}\| \, \|T\|$$

of the norms of the projections $I - P$ and P, where I is the identity operator on Y.

4. Strong Unicity for L_p-Spaces

Let (S, Σ, μ) be a positive measure space. In the present section we shall use Theorem 1.1 to deduce the existence of strongly unique best approximations in the space $Y = L_p = L_p(S, \Sigma, \mu)$ of all μ-measurable real valued functions (equivalence classes) y on S such that

$$\|y\| = \|y\|_p = \left[\int_S |y(s)|^p \, \mu(ds) \right]^{1/p} < \infty, \qquad 2 \leqslant p < \infty.$$

We first establish two auxiliary lemmas.

Lemma 4.1. *If* $0 \leqslant u_i \leqslant m$ $(i = 1, 2,\ m > 0)$ *then*

$$|u_1^p - u_2^p| \leqslant p m^{p-1} |u_1 - u_2|, \qquad p \geqslant 1.$$

Proof. Apply the mean value theorem to the function $f(u) = u^p$. ∎

Lemma 4.2. *If* $t \in [0, 1]$, $u, v \in \mathbb{R}$, *and* $2 \leqslant p < \infty$ *then*

$$t\,|u + v|^p + (1 - t)\,|u|^p - |u + tv|^p \geqslant w(t)\,|v|^p, \tag{4.1}$$

where

$$w(t) = 2^{2-p}[t(1-t)^p + (1-t)\,t^p].$$

Proof. If $v = 0$ or $p = 2$, then the proof is trivial. Otherwise, let us denote $u = -s \cdot v$, $s \in \mathbb{R}$. Then the inequality (4.1) is equivalent to the inequality

$$f(t, s) \geqslant 0; \qquad t \in [0, 1], \qquad s \in \mathbb{R}, \tag{4.2}$$

where

$$f(t, s) = t\,|1 - s|^p + (1 - t)\,|s|^p - |s - t|^p - w(t).$$

This inequality is trivial for $t = 0, 1, s$. Moreover, note that $f(t, s) = f(1 - t,$

$1-s$). Hence it is sufficient to prove the inequality (4.2) only for s in the intervals

$$A_t = \{s \in \mathbb{R}: s > t \text{ and } 0 < t < 1\}.$$

For this purpose we define the functions F_t on A_t by

$$F_t(s) = -t\,\operatorname{sign}(1-s)[|1-s|/(s-t)]^{p-1} + (1-t)(s/(s-t))^{p-1} - 1.$$

Since

$$F_t'(s) = (p-1)\,t(1-t)(|1-s|^{p-2} - s^{p-2})/(s-t)^p,$$

it follows that $F_t(s)$ strictly decreases (increases) for $s > \max(t, \frac{1}{2})$ ($t < s \leqslant \frac{1}{2}$, respectively). Hence

$$\frac{\partial f}{\partial s} = p(s-t)^{p-1}\,F_t(s) > \lim_{s \to +\infty} F_t(s) = 0$$

for all $s > \frac{1}{2}$ in A_t. If $t \geqslant \frac{1}{2}$, then $\partial f/\partial s > 0$ implies $f(t, s)$ is increasing, so $f(t, s) \geqslant f(t, t) \geqslant 0$. Further, by the fact that

$$\frac{\partial f}{\partial s}(t, t) < 0 < \frac{\partial f}{\partial s}\left(t, \frac{1}{2}\right), \qquad 0 < t < \frac{1}{2},$$

we conclude that there exists a unique $s_t \in (t, \frac{1}{2})$ such that

$$\frac{\partial f}{\partial s}(t, s_t) = -t(1-s_t)^{p-1} + (1-t)\,s_t^{p-1} - (s_t - t)^{p-1} = 0, \quad 0 < t < \tfrac{1}{2}. \tag{4.3}$$

Therefore, we obtain

$$\begin{aligned} f(t, s) \geqslant f(t, s_t) &= t(1-t)\{[s_t^{p-1} + (1-s_t)^{p-1}] \\ &\quad - 2^{2-p}[t^{p-1} + (1-t)^{p-1}]\} \\ &> t(1-t)\{2^{2-p} - 2^{2-p} \cdot 1\} = 0 \end{aligned}$$

for all s in A_t, $t \in (0, 1)$. This completes the proof. ▌

Let us note that Lemma 4.2 is not true for $1 \leqslant p < 2$. Indeed, by the L'Hôpital rule, we have

$$\begin{aligned} \lim_{s \to +\infty} [f(t, s) + w(t)] &= t \lim_{s \to +\infty} s^{p-2}[(1-s^{-1})^{p-2} \\ &\quad - t(1-t/s)^{p-2}] = 0, \end{aligned}$$

where $f(t, s)$ and $w(t)$ are as in (4.2) and (4.1), respectively.

THEOREM 4.1. *Let X be a linear closed subspace of L_p, $p \geqslant 2$. Then for a function y in L_p there exists a unique function z in X such that*

$$\|y-z\|^p \leqslant \|y-x\|^p - 2^{2-p}\|z-x\|^p \tag{4.4}$$

for all x in X.

Proof. Let us define

$$f(x) = \|y-x\|^p, \qquad a(s) = p2^{2-p}s^{p-1},$$

and

$$d(r) = p(r + \|y\|)^{p-1}.$$

Then, by using notations from Theorem 1.1, we have

$$b(s) = 2^{2-p}s^p \qquad \text{and} \qquad c(t, s) = w(t)\, s^p,$$

where $w(t)$ is as in Lemma 4.2. Now, if $x_i \in X$ ($\|x_i\| \leqslant r$, $i = 1, 2$) then we have $u_i := \|y - x_i\| \leqslant r + \|y\|$. Hence by Lemma 4.1 we obtain

$$\begin{aligned} |f(x_1) - f(x_2)| = |u_1^p - u_2^p| &\leqslant d(r)\, |\, \|x_1 - y\| - \|y - x_2\|\, | \\ &\leqslant d(r)\, \|x_1 - x_2\|. \end{aligned}$$

Thus the condition (i) in Theorem 1.1 is satisfied. In order to verify the condition (ii) in Theorem 1.1, we put $u = y(s) - x(s)$ and $v = -h(s)$ into the inequality (4.1) and integrate both sides. This gives the inequality

$$\begin{aligned} g(t; x, h) = t\, \|y - x - h\|^p + (1-t)\, \|y - x\|^p \\ - \|y - x - th\|^p \geqslant c(t, \|h\|), \end{aligned}$$

where t and x, h are arbitrary elements of the interval $(0, 1)$ and the subspace X, respectively. This completes the proof of the condition (ii). Finally, by applying Theorem 1.1, we immediately obtain (4.4). ∎

This theorem says that there exists a strongly unique best approximation z in X to every y in L_p $(p \geqslant 2)$ in the sense of Definition 1.1, $\varphi(s) = s^p$ and $K = 2^{2-p}$. Clearly, the function z is the unique best approximation in X to the function y. When $p = 2$, then these results coincide with the corresponding results obtained in Section 2. Now, let the projection $P = P_p$ of L_p $(p \geqslant 2)$ onto X be defined by $Py = z$. In general, this is a linear projection only for $p = 2$. If we put $x = 0$ into (4.4) then we directly obtain the corollary.

COROLLARY 4.1. *For every y in L_p $(p \geqslant 2)$ we have*

$$\|Py\|^p \leqslant 2^{p-2}(\|y\|^p - \|y - Py\|^p),$$

$$\|Py\| \leqslant 2^{1-2/p}\|y\| \qquad \textit{and} \qquad \|y - Py\| \leqslant \|y\|.$$

When $p=2$, then the second inequality in this corollary implies that the linear projection P satisfies a Lipschitz condition of order 1 with the constant 1. In the case $p>2$, we show that P satisfies a local Lipschitz condition of the order $1/p$.

COROLLARY 4.2. *For every y_1, y_2 in a ball $B(r) = \{y \in L_p : \|y\| \leqslant r\}$ of L_p $(p \geqslant 2)$ we have*

$$\|Py_1 - Py_2\| \leqslant k_r \|y_1 - y_2\|^{1/p},$$

where $k_r < 6\sqrt{2}\, r^{1-1/p}$.

Proof. Let us put $y = y_1$, $x = Py_2$ and $y = y_2$, $x = Py_1$ into the inequality (4.4), and sum up the obtained inequalities. Then, by applying Lemma 4.1, we derive

$$\begin{aligned} 2^{2-p}\|Py_1 - Py_2\|^p &\leqslant \tfrac{1}{2}(\|y_1 - Py_2\|^p - \|y_2 - Py_2\|^p \\ &\quad + \tfrac{1}{2}(\|y_2 - Py_1\|^p - \|y_1 - Py_1\|^p) \\ &\leqslant p[r(1 + 2^{1-2/p})]^{p-1}\|y_1 - y_2\| \end{aligned}$$

for any y_1, y_2 in $B(r)$. Hence the desired inequality follows immediately. ∎

5. STRONG UNICITY FOR SPLINES IN L_p-SPACES

In the present section we briefly discuss some properties of spline approximation with respect to a linear bounded operator T on a real Banach space Y into the space $Y_1 = L_p = L_p(S, \Sigma, \mu)$ $(p>2)$, which is defined as in Section 3. Here it is assumed that X, X_1, T_0, and $\sigma = y - z$ have the same meaning as in Section 3. Thus the only difference between spline approximations considered in Section 3 and spline approximations of this section consists in replacing the Hilbert space Y_1 by an L_p-space, $p>2$.

THEOREM 5.1. *For an element y in Y there exists a strongly unique spline approximation $\sigma = y - z$ $(z \in X)$, i.e.,*

$$\|T\sigma\|^p \leqslant \|Ty - Tx\|^p - 2^{2-p}\|T_0^{-1}\|^{-p}\|z - x\|^p, \qquad p>2,$$

for all x in X.

Proof. Let us replace $a(s)$, $b(s)$, $c(t, s)$, $d(r)$, and $f(x) = \|y - x\|^p$ in the proof of Theorem 4.1 by $ka(s)$, $kb(s)$, $kc(t, s)$ $(k = \|T_0^{-1}\|^{-p})$, $\|T\|\ d(r)$, and $f(x) = \|Ty - Tx\|^p$, respectively.

Since $\|Th\| \geqslant \|T_0^{-1}\|^{-1}\ \|h\|$ for any $h \in X$ (see Section 3), we can now repeat mutatis mutandis the proof of Theorem 4.1 to complete the proof of this theorem. ∎

From this theorem we immediately conclude that the spline projection $Py = \sigma$, $y \in Y$, has the following properties.

COROLLARY 5.1. *For every y in Y we have*

$$\|y - Py\|^p \leqslant 2^{p-2}\ \|T_0^{-1}\|^p(\|Ty\|^p - \|TPy\|^p),$$

$$\|y - Py\| \leqslant M \qquad \text{and} \qquad \|Py\| \leqslant \|y\| + M,$$

where

$$M = 2^{1-2/p}\ \|T_0^{-1}\|\ \|Ty\|.$$

6. Strong Unicity and Invariant Approximation

Let F be a nonexpansive map of a Banach space Y into itself, i.e.,

$$\|Fy_1 - Fy_2\| \leqslant \|y_1 - y_2\|$$

for any y_i $(i = 1, 2)$ in Y. Following Meinardus [10], we can introduce a notion of invariant approximation as follows.

DEFINITION 6.1. A best approximation z in X to an element y in Y such that $Fy = y$ is called an invariant approximation in X to y if $Fz = z$.

In some cases, by using an appropriately chosen fixed point theorem, one can prove the invariance of a best approximation [10, 16, 17]. On the other hand, one can easily notice that there is a direct link between the notions of invariant approximation and strongly unique best approximation in the sense of Definition 1.1. More precisely, we have

THEOREM 6.1. *Let z be a strongly unique best approximation in X to $y \in Y$ such that $Fy = y$. Then z is an invariant approximation in X to y.*

Proof. By Definition 1.1 we have

$$\begin{aligned} K\varphi(\|z - Fz\|) &\leqslant \varphi(\|Fy - Fz\|) - \varphi(\|y - z\|) \\ &\leqslant \varphi(\|y - z\|) - \varphi(\|y - z\|) = 0. \end{aligned}$$

Since $\varphi(0)=0$, $\varphi(s)>0$ for $s>0$ and $K>0$, it follows that $\|z-Fz\|=0$. This completes the proof. ∎

In particular, this theorem implies that a best approximation z in X to $y \in Y$ considered in Sections 2 and 4 is invariant. When the map F is linear and $TF=FT$, a reasoning similar to that in Theorem 6.1 shows that the spline approximations of Sections 3 and 5 are also invariant.

Note added in proof. We have shown, jointly with B. Prus, that the best L_p-approximations, $1<p<2$, are strongly unique in the sense of Definition 1.1 with respect to the function $\varphi(s)=s^2$.

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